

# About embedded quarters and points at infinity in the hyperbolic plane

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## Abstract

In this paper, we prove two results. First, there is a family of sequences of embedded quarters of the hyperbolic plane such that any sequence converges to a limit which is an end of the hyperbolic plane. Second, there is no algorithm which would allow us to check whether two given ends are equal or not.

**Keywords:** hyperbolic plane, pentagrid, sequence of quarters, ends of the hyperbolic plane

## 1 Introduction

This study takes place in hyperbolic geometry, in a specific tiling of the hyperbolic plane, the tessellation  $\{5, 4\}$  which I called the **pentagrid**, see [2].

Fix such a tessellation. Denote by  $a$  the length of a side of a tile of the tessellation. In this tiling, we call **quarter**, a subset of the tiling which is the intersection of two half-planes whose lines support consecutive edges of a pentagon  $P$  of the tessellation. This pentagon is called the **head** of the quarter and the common point of the lines delimiting the half-planes is called the **vertex** of the quarter. Note that the quarter is delimited by two rays issued from the vertex and supported by the above mentioned lines. These rays are also the **border** of the quarter.

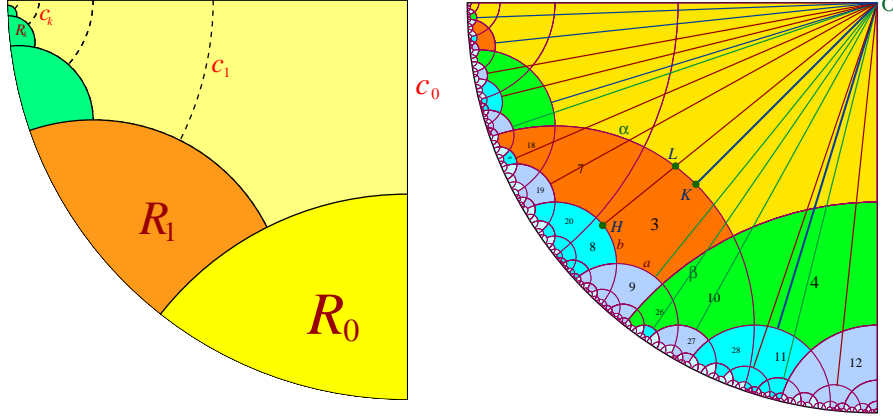
In this paper we are interested by sequences of quarters such that each term of the sequence is included in the next one. We shall show that the vertices of such quarters tend to a limit. To this aim, Section 3 fixes the notion of neighbourhood for a point at infinity. Section 4 studies simple properties of included quarters. But before, we had to establish specific projection properties

of the pentagrid in Section 2. Section 5 proves that a sequence of embedded quarters has a limit and Section 6 shows two results of undecidability concerning points at infinity.

## 2 Prolegomenon: the cornucopia representation

We fix  $O$  a point of the hyperbolic plane and two orthogonal rays issued from  $O$ :  $p$  and  $q$ . We may assume that, counter-clockwise turning around  $O$ ,  $p$  comes before  $q$ . We say that  $p$  is **horizontal** and that  $q$  is **vertical**. The rays  $p$  and  $q$  constitute the border of a **quarter** of the plane,  $\mathcal{Q}$ . Such a quarter can also be viewed as the intersection of two half-planes whose borders are perpendicular.

First, let us fix notations. Consider a pentagon  $P$ . Counter-clockwise and consecutively number the sides of  $P$  by  $i$  with  $i \in [1..5]$ . Denote by  $\ell_i$  be the line which supports the side  $i$  and let  $A, B, C, D$  and  $E$  be the vertices of  $P$ , counter-clockwise labelled in this way, with  $A, E$  belonging to both sides 5 and 1, sides 5 and 4 respectively. Each line  $\ell_i$  defines two half-planes  $H_i$  and  $\neg H_i$ . Let  $H_i$  denote the half-plane which contains  $P$ . Call **lower strip** of  $P$  the region which is defined by  $H_1 \cap H_4 \cap \neg H_5$ . In the lemmas of the paper, we shall speak of the side  $i$  of a pentagon, having in mind a numbering as the one we already considered for  $P$ , and we shall always remind which side is side 1 in order to avoid ambiguities. Note that sides 1 and 4 are opposite and that  $\ell_5$  is the common perpendicular of  $\ell_1$  and  $\ell_4$ .



**Figure 1** *To left: the cornucopia.  
To right: proving properties of the cornucopia.*

Let  $\{P_n\}_{n \in \mathbb{N}}$  denote a sequence of pentagons lying in  $\mathcal{Q}$  such that any  $P_n$  has an edge contained in  $p$ , such that  $P_0$  has  $O$  as a vertex and that its edges meeting at  $O$  are contained in  $p$  and  $q$ , see Figure 1, and such that for any  $n$ ,  $P_n$  and  $P_{n+1}$  have a common side: it is both the side 1 of  $P_n$  and the side 4 of  $P_{n+1}$ . The complement of the  $P_n$ 's in  $\mathcal{Q}$  can be represented as a union of quarters  $R_n$  as illustrated by the left-hand side picture of Figure 1. We call **cornucopia of  $\mathcal{Q}$**  the union of the  $P_n$ 's. The  $R_n$ 's are defined as follows:  $R_0$  is bordered by  $q$  and by the line which support the side 3 of  $P_0$ ,  $R_{n+1}$  is bordered by the line which supports the side 3 of  $P_n$  and by the line which supports the

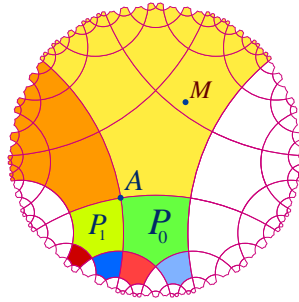
side 2 of  $P_n$ . Remember that in a pentagon, sides 2 and 3 are perpendicular at the point where they meet.

The quarters  $R_n$  can be defined in another way:  $R_0$  is the image of  $\mathcal{Q}$  by the shift  $\tau_0$  along  $q$  of amplitude  $a$ . Note that  $\tau_0$  transforms  $O$  in the other vertex of the side 4 of  $P_0$  which lies in  $q$ . Note that the shift  $\tau$  along  $p$  of amplitude  $a$  transforms the side 1 of  $P_n$  into the side 1 of  $P_{n+1}$ . Now, let  $\mathcal{Q}_{n+1}$  be the image of  $\mathcal{Q}_n$  by  $\tau$ , putting  $\mathcal{Q}_0 = \mathcal{Q}$ . Then,  $R_{n+1}$  is the image of  $R_n$  under  $\tau$ . Note that  $R_{n+1}$  is also the image of  $\mathcal{Q}_n$  by the shift  $\tau_n$  along the side 1 of  $P_n$  of amplitude  $a$ . Note that  $\tau_n$  translates this decomposition of  $\mathcal{Q}$  into each quarter  $R_{n+1}$ . Consider the recursive iteration of this decomposition in all new quarters generated in this way. We say that the regions  $R_m$  belong to the first generation, so that the shift of the decomposition of  $\mathcal{Q}_m$  in each of them by  $\tau_m$  defines the second generation. In a similar way, the generation  $n+1$  is obtained from the generation  $n$ . The decomposition of each region into the cornucopia and its complement constitute the **cornucopia decomposition** of  $\mathcal{Q}$ .

Presently, we wish to give a better algorithmic representation of the cornucopia decomposition of  $\mathcal{Q}$  which will allow us to prove interesting properties.

**Lemma 1** *Let  $R_2$  and  $R_3$  be the pentagons obtained from  $P$  by reflection in its sides 2 and 3 respectively. Define the side 5 of  $R_2, R_3$  to be the side 2, 3 of  $P$  respectively. Then the lower strip of  $R_2, R_3$  respectively, contains the lower strip of  $P$ .*

Proof. Remember that the lower strip  $\mathcal{S}$  of  $P$  is defined as  $H_1 \cap H_4 \cap \neg H_5$ . Note that  $R_2, R_3$  is also the shift  $\tau_1, \tau_4$  respectively of  $P$  along the side 1, 4, respectively, of  $P$  of amplitude  $a$ , see Figure 2 where  $P_0$  plays the role of  $P$ . Denote by  $\mathcal{S}_2, \mathcal{S}_3$  the strip of  $R_2, R_3$  respectively. Then,  $\mathcal{S}_i = H_1^{\tau_{j_i}} \cap H_4^{\tau_{j_i}} \cap \neg H_5^{\tau_{j_i}}$ , with  $i \in \{2, 3\}$ ,  $j_2 = 1$  and  $j_3 = 4$ . We have that  $H_1^{\tau_1} = H_1$ . Now,  $\tau_1$  can be decomposed into the reflection  $\beta$  in the bisector of side 1 followed by the reflection  $\rho$  in side 2. Now,  $\beta$  transforms  $\ell_4$  into  $\ell_3$  and  $\rho$  leaves  $\ell_3$  globally invariant, so that  $H_4^{\tau_1} = H_3$ . We have too that  $\beta$  transforms  $\ell_5$  into  $\ell_2$  and  $\ell_2$  is invariant under  $\rho$ . Consequently,  $(\neg H_5)^{\tau_1} = H_2$ . Now, a product of two reflections in axes which are perpendicular to  $\ell_1$  shows that  $\mathcal{S} \subset H_3 \cap H_2$ . Hence,  $\mathcal{S} \subset \mathcal{S}_2$ . Similarly,  $H_4^{\tau_4} = H_4$ ,  $H_1^{\tau_4} = H_2$  and  $(\neg H_5)^{\tau_4} = H_3$ , so that we obtain that  $\mathcal{S} \subset \mathcal{S}_3$ .  $\square$



**Figure 2** *Illustration of the proofs of Lemma 1 and Lemma 2.*

**Lemma 2** *Consider the pentagon  $P$ . Let  $\tau$  be the shift along the side 5 of  $P$  of amplitude  $a$ , transforming  $\ell_4$  into  $\ell_1$ . Let  $Q$  be the image of  $P$  under  $\tau$ . Let  $\mathcal{S}_2$ ,*

$S_3$  be the image of  $Q$  by reflection in its sides 2, 3 respectively. Let the side 5 of  $S_2, S_3$  be the side 2, 3 of  $Q$  respectively. Then the lower strip of  $S_2$  contains that of  $P$ , but the strip of  $S_3$  does not meet that of  $P$ , except on the line  $\ell_4$  of  $P$ .

Proof. Denote by  $\sigma_1, \sigma_4$  the shift of amplitude  $a$  along the line 1, 4 of  $Q$  respectively, which transforms the side 5 of  $Q$  into its side 2, 3 respectively. Denote by  $i_Q$  the side  $i$  of  $Q$ . Denote by  $\mathcal{T}_2, \mathcal{T}_3$  the lower strip of  $S_2, S_3$  respectively. Repeating the proof of Lemma 1, we obtain that  $\mathcal{T}_i = H_{1_Q}^{\sigma_{j_i}} \cap H_{4_Q}^{\sigma_{j_i}} \cap (\neg H_{5_Q})^{\sigma_{j_i}}$ , with  $i \in \{2, 3\}$  and  $j_2 = 1$  and  $j_3 = 4$ . Repeating the same argument, we get that  $\mathcal{T}_2 = H_{1_Q} \cap H_{3_Q} \cap (\neg H_{2_Q})$ . Now,  $H_1 \subset H_{1_Q}$  as  $H_{1_Q} = H_1^\tau$ . This is obtained by decomposing  $\tau$  into the reflection  $\gamma$  in the bisector of side 5 followed by a the reflection  $\rho_1$  in the side 1. similarly, we have that  $H_4 \subset H_{4_Q}^\tau = H_{4_Q}$ . At last, note that  $H_{5_Q} = H_5$ , so that  $(\neg H_{5_Q})^{\sigma_1} = \neg H_{2_Q}$  and  $\neg H_5 \subset \neg H_{2_Q}$ . From this we get that  $\mathcal{S} \subset \mathcal{T}_2$ . For  $S_3$ , note that  $H_{4_Q} \cap H_1 \subset \ell_1$ . Now,  $\ell_{4_Q} = \ell_1$ . Accordingly, as  $\mathcal{T}_3 \subset H_{4_Q} = \neg H_1$ ,  $\mathcal{T}_3$  cannot meet  $\mathcal{S}$ .  $\square$

**Lemma 3** *Consider a pentagon  $P$  with its sides, their support and its vertices labelled as above indicated. Let  $M$  be a point in  $\mathcal{S}$ , the lower strip of  $P$ . Let  $K$  be the orthogonal projection of  $M$  on  $\ell_5$ . Then  $K$  is in side 5. If  $M$  is in  $\ell_1$  or in  $\ell_4$ , then  $M = A$  or  $M = E$  respectively.*

Proof. Let  $H_1, H_4$  be the half-plane defined by  $\ell_1, \ell_4$  respectively which contains  $P$ . If  $K \notin \mathcal{S}$ , then, by construction of  $H_1$  and  $H_4$  we have  $K \notin H_1$  or  $K \notin H_4$ . Assume that  $K \notin H_1$ . Then,  $PK$  cuts  $\ell_4$  in  $T$ . Whether  $T = E$  or  $T \neq E$ , from  $T$  we have two distinct perpendiculars to  $\ell_5$  which is impossible. A similar argument proves that  $K$  cannot be in  $H_1$ .  $\square$

**Lemma 4** *Let  $P$  be a pentagon with the same labelling as in Lemma 3. Let  $M$  belong to the lower strip of  $P$ . Let  $K, F$  and  $G$  be the orthogonal projection of  $M$  on  $\ell_5, \ell_2$  and  $\ell_3$  respectively. Then,  $F$  belongs to side 2,  $G$  belongs to side 3,  $MF$  cuts  $\ell_5$  in the open segment  $]AK[$  and  $MG$  cuts  $\ell_5$  in the open segment  $]KE[$ . Note that if  $M$  belongs to  $\ell_1, \ell_4$ , then  $K$  and  $F, G$  respectively also belong to  $\ell_1, \ell_4$  respectively, and the conclusion for  $G, F$  respectively still holds.*

Proof. From Lemma 3,  $K$  belongs to the side 5 of  $P$ . Let  $U$  and  $V$  be the reflections of  $P$  in  $\ell_2$  and  $\ell_3$  respectively. From Lemma 1, the lower strip of  $P$  is both contained in the lower strip of  $U$  and in that of  $V$ . Accordingly,  $F$  belongs to side 2 and  $G$  belongs to side 3. As  $M$  is not in the same side of  $\ell_5$  as  $P$ ,  $MF, MG$  cuts side 5 in  $R, S$  respectively: Note that, as  $\ell_4$  is the common perpendicular to  $\ell_5$  and  $\ell_3$ , and as  $MK$  is perpendicular to the side 5 of  $P$ ,  $K \neq E, K \neq A, F \neq D$  and  $G \neq B$ . Also note that  $R \neq K$  and  $S \neq K$ . Otherwise, if  $R$  or  $S$  would coincide with  $K$ ,  $KFBA$  or  $DEKG$  respectively would be a rectangle, which is impossible. Now, by construction,  $RFBA$  is a Lambert quadrangle, so that  $(RA, RF)$  must be acute. Clearly,  $(RM, RK) = (RA, RF)$ . As  $MK$  is perpendicular to the side 5 of  $P$ ,  $(RM, RK)$  is an acute angle, so that we must have  $]ARK[$ :  $R$  is inside  $AK$ . A similar argument with the Lambert quadrangle  $EDGS$  shows us that  $S$  is in  $]KE[$ . The case when  $M$  is on  $\ell_1$  or on  $\ell_4$  is obvious.  $\square$

Let us go back to the cornucopia decomposition of  $\mathcal{Q}$ .

**Lemma 5** Consider the cornucopia decomposition of  $\mathcal{Q}$ . Consider a region  $R$  of the generation  $n$ : let  $P_i$ 's be the pentagons of the cornucopia of  $R$ , and let  $R_i$ 's be the regions of the generation  $n+1$  inside  $R$ , both sequences of objects being numbered as in the cornucopia of  $\mathcal{Q}$ . Then, the head of the region  $R_0$  inside  $R$  is the image of the head of  $P_0$  by the shift along the side 4 of  $P_0$  with an amplitude of  $a$ , and the head of the region  $R_i$  inside  $R$  with  $i \geq 1$  is the image of  $P_{i-1}$  under the shift along the side 1 of  $P_{i-1}$  with an amplitude of  $a$ . Under these shifts, the correspondence between the sides/lines of  $P_i$  and those of the head of  $R_{i+1}$  as well as between the sides/lines of  $P_0$  and those of the head of  $R_0$  is given by Table 1.

**Table 1** The numbers concern the lines when they are identical in  $R_{i+1}$  or  $R_i$  with those of  $P_i$ .

$P_i$	1	2	3	4	5
$R_{i+1}$	1	5	4		
$R_0$		1	5	4	

Proof. The line in Table 1 associated to  $R_0$  is a corollary of Lemma 1. For the regions  $R_i$  with  $i \geq 1$ , this is a corollary of Lemma 2. Remember that the shifts described in the statement of the lemma keep the orientation of the numbering invariant and that due to the definition of the shifts, a side  $i$  is transformed into a side  $i$  under a shift along the support of the former side  $i$  for  $i \in \{1, 4, 5\}$ .  $\square$

**Corollary 1** Let  $R$  be a region in the cornucopia decomposition of  $\mathcal{Q}$ . Let  $T$  be the head of  $R$  and  $\ell$  be the line which supports the side 5 of  $T$ . Then, the half-plane defined by  $\ell$  which does not contain  $T$  contains  $O$ .

Proof. This is a corollary of Lemma 5. We know that the head of the region is delimited by its side 5. From Lemma 3,  $O$  belongs to the lower strip of  $T_0$  and of  $T_1$ , the heads of the region  $R_0$  and  $R_1$  of generation 1. Lemma 2 extends this property to all the other regions  $R_i$  of generation 1.

Assume that the property is true for the generation  $n$ . Consider a region  $R^n$  of the generation  $n$ . Let  $T$  be its head and let  $H$  be the half-plane defined by the support of the side 5 of  $T$  which does not contain  $O$ . Then, the heads of the regions  $R_0$  and  $R_1$  of the generation  $n+1$  are contained in  $H$ , so that Lemma 5 applied to  $T$  says that  $O$  is also in the lower strip of the heads of the regions  $R_0^{n+1}$  and  $R_1^{n+1}$  of the generation  $n+1$ : consequently, the property also holds for these two regions. The shift along the side 5 of  $T$  of amplitude  $a$  which transforms the side 4 of  $T$  into its side 1 satisfies the hypothesis of Lemma 2. By induction, the lemma allows us to extend the property from the region  $R_i^{n+1}$  with  $i \geq 1$  to the regions  $R_{i+1}^{n+1}$ . Accordingly, the property is true for all regions of the generation  $n+1$ . This completes the proof of the corollary.  $\square$

**Corollary 2** Consider a region  $R$  of the cornucopia decomposition of  $\mathcal{Q}$ . Then  $O$  is in the lower strip of the head of  $R$ . For another pentagon  $Q$  of the cornucopia of  $R$ ,  $O$  is in the lower strip of the pentagon which is the image of  $Q$  under the shift along its side 1, going from the border of  $R$  to the side 2 of  $Q$ .

Proof. This is also a consequence of the proof given for Corollary 1.  $\square$

We arrive to the key property of this section.

**Lemma 6** In  $\mathcal{Q}$ , the distance from  $O$  to a region of the generation  $n$  is at least  $n \cdot a$ .

We need a preliminary result:

**Lemma 7** *For each region  $R$  in the cornucopia decomposition of  $\mathcal{Q}$ , the orthogonal projection of  $O$  on the border of  $R$  occurs on the side 5 of its head, ends of the side excepted when  $q$  is not a border of the region.*

Proof. This is a corollary of Corollary 2 and of Lemma 4 and of the fact that the projection of  $O$  on a region  $R$  is the same as its projection on the head  $H$  of  $R$ : the projection is also the projection of  $O$  on the line  $\ell$  supporting the side 5 of  $H$ . Accordingly, all points in the half-plane defined by  $\ell$  which does not contain  $O$  are further from  $O$  than its projection on  $\ell$ .  $\square$

Proof of Lemma 6. Note that the result is true for generation 1. The cornucopia of  $\mathcal{Q}$  has a complex border: it is  $p$  which contains the side 5 of all pentagons contained in the cornucopia. Another infinite part of the border consists of the sides 5 of the heads of the regions of generation 1. As the pentagons  $P_i$  with  $i \geq 1$  are outside the half-plane defined by the side 1 of  $P_0$  which does not contain  $O$ , the distance of each  $P_i$  to  $O$  is at least  $a$ . In particular, this is the case for  $OK_n$  where  $K_n$  is the orthogonal projection of  $O$  on the border of  $R_n$ ,  $n \geq 0$ . Now, as  $R_i$  is contained in the half-plane defined by the side 5 of its head containing its head, the distance from  $O$  to  $R_n$  is at least  $OK_n$ , so that it is at least  $a$ .

Assume that the result is true for the generation  $n$ . Consider a region  $R$  of the generation  $n$  and consider a region  $R^1$  of the generation  $n+1$  contained in  $R^1$ . The head of  $R^1$  is obtained from a pentagon  $P_i$  of the cornucopia of  $R$ . Now, from Lemma 7, the orthogonal projection  $K^1$  of  $O$  on  $R^1$  occurs on the head of  $R^1$ . Let  $K$  be the orthogonal projection of  $O$  on  $R$ . Unless  $R^1$  is the region 0 of  $R$ , the head of  $R^1$  is obtained from the head of  $R$  by a shift along the side 1 of  $R$ . From Lemma 4, we have that  $K^1$  is in the side 2 of the head of  $K$ : informally,  $OK^1$  is to the left of  $OK$ . Let  $OK^1$  cuts the side 5 of the head of  $R$  at  $L$ . As the quadrangle  $ABK^1L$  is a Lambert quadrangle, remember that  $AB$  is the side 1 of the head of  $R$ , the angle  $(LA, LK^1)$  is acute, so that  $LK^1 > a$ . On another hand,  $OL > OK$  as  $L$  is on the side 5 of the head of  $R$  and as  $L \neq K$ . Accordingly,  $OK^1 = OL + LK^1 > n \cdot a + a$ . If  $R^1$  is the region 0, then  $K^1$  is on the side 3 of the head of  $R$ . Now, we consider the quadrangle  $LK^1DE$  which is also a Lambert quadrangle, so that the same estimates can be performed, leading us to the same conclusion. And so, the property is true for the regions of the generation  $n+1$ .  $\square$

Lemma 6 has a very important corollary which we establish now, although it is not tightly connected to the topic of this paper.

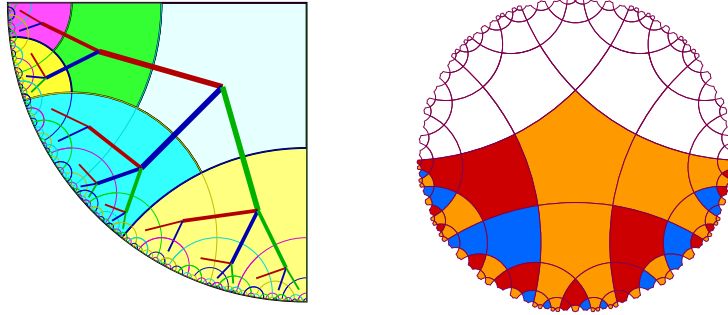
The left-hand side picture of Figure 3 illustrates the bijection between the restriction of the pentagrid to  $\mathcal{Q}$  with a tree we called the Fibonacci tree, see [1, 2]. The name of the tree comes from the fact that the number of nodes of the tree which are at the same distance  $d$  from its root in term of crossed tiles is  $f_{2d+1}$  where  $\{f_n\}_{n \in \mathbb{N}}$  is the Fibonacci sequence with  $f_0 = f_1 = 1$ . In [2], we remember the proof of the property already mentioned in [1, 4] that the restriction of the pentagrid to  $\mathcal{Q}$  is in bijection with a tree which we called the Fibonacci tree: The tree can be constructed by the infinite iteration of two rules we can formulate as  $W \rightarrow BWW$  and  $B \rightarrow BW$ ,  $B$  denoting the nodes which have two sons and  $W$  denoting those which have three of them, the root of the tree being a  $W$ -node. We can state the following result:

**Theorem 1** (see [4, 1, 2]) *The Fibonacci tree is in bijection with the restriction of the pentagrid to  $\mathcal{Q}$ .*

*Proof.* The proof of the injection is easy: it is enough to note that the sons of a node  $\nu$  are obtained by the reflection of the tile  $T$  associated to  $\nu$  in two or three different sides of  $T$ .

For the surjection, we have to prove that any point of  $\mathcal{Q}$  belongs to a tile of the pentagrid restricted to  $\mathcal{Q}$ . Using the cornucopia decomposition, it is rather easy. Let  $M$  be a point of  $\mathcal{Q}$ . If  $M$  belongs to the cornucopia of  $\mathcal{Q}$ , it belongs to some  $P_i$  and we are done. If this is not the case, it belongs to some  $R^1$  of generation 1. In  $R^1$  we repeat the same argument: either  $M$  belongs to the cornucopia of  $R^1$  and we find a pentagon of the tiling containing  $M$ , or we find that  $M$  belongs to some region  $R^2$  of generation 2. As from Lemma 6, the distance from  $O$  to a region of the generation  $n$  is at least  $n \cdot a$ , we can find an  $m$  such that  $m \cdot a > OM$ , so that necessarily,  $M$  belongs to the cornucopia of a region  $R^k$  of the generation  $k$  with  $k < m$ . Eventually,  $M$  belongs to some pentagon of the tiling.

It is not difficult to see that the pentagons of the cornucopia decomposition are those of the Fibonacci tree: the cornucopia of  $\mathcal{Q}$  corresponds to the leftmost branch of the Fibonacci tree. Its regions  $R_0$  and  $R_1$  have the  $W$ -sons of the root for their respective heads. Now, in each region, the cornucopia is the leftmost branch of the sub-tree rooted at the node corresponding to the head. Note that the heads of the regions are the white nodes of the tree, the root being the head of  $\mathcal{Q}$ . This is illustrated by the right-hand side picture of Figure 3.  $\square$



**Figure 3** *To left: the bijection between the tree and the quarter. A red arrow leads to a **black node**, the others lead to a **white** one. The root of the tree is considered as a white node.*

*To right: correspondence between the cornucopias of the decomposition of  $\mathcal{Q}$  and the black nodes of the Fibonacci tree. The black nodes are the tiles in blue and in red. The other coloured tiles are white nodes.*

### 3 Convergence at infinity

In the following sections, we shall have to deal with sequences of points which are converging to infinity. Convergence in the hyperbolic plane is easy and we can rely on Poincaré's disc model as far as topology only is concerned. To study points at infinity, we have to resist to the use of Poincaré's disc model: it fairly

represents what Hilbert called ends in the hyperbolic plane, but there is always the danger that the Euclidean intuition plays some bad trick on us. In order to define convergence to infinity, we have to justify that the notion of convergence to a point of the border in Poincaré's disc model turns out to be valid.

Consider the same fixed point  $O$  of the hyperbolic plane and the same quarter  $\mathcal{Q}$  whose vertex is  $O$  which were defined in Section 2 and consider a sequence  $\sigma$  of points  $\{x_n\}_{n \in \mathbb{N}}$  of the hyperbolic plane such that  $x_n \in \mathcal{Q}$  for all  $n$  and that  $Ox_n$  tends to infinity as  $n$  tends to infinity. Say that a **neighbourhood of infinity** for  $\sigma$  is a half-plane  $H$  defined by a line  $\ell$  such that  $\mathbb{H}^2 \setminus H$  contains finitely many points of  $\sigma$  only.

Consider a point at infinity  $\alpha$  and a line  $\ell$  which does not pass through  $\alpha$ . The line defines two half-planes:  $H_1$  and  $H_2$ . In one of them, say  $H_2$ , any line contained in the half-plane does not pass through  $\alpha$ . In the other, there are such lines : for any point  $M$  of  $\ell$  there is a unique line which passes through  $M$  and through  $\alpha$ . In our sequel we say that  $H_1$  is the half-plane defined by  $\ell$  which **touches**  $\alpha$  and that  $H_2$  is the one which does **not touch**  $\alpha$ .

**Lemma 8** *Let  $\ell$  be a line of the hyperbolic plane and let  $H$  be the half-plane delimited by  $\ell$  which does not contain  $O$ . Let  $K$  be the orthogonal projection of  $O$  on  $\ell$ . Let  $\delta_1$  and  $\delta_2$  be the ray issued from  $O$  which are parallel to  $\ell$ . Then  $(\delta_1, \delta_2)$  tends to zero as  $OK$  tends to infinity and conversely.*

Proof. By construction, as  $OK \perp \ell$ ,  $(\delta_1, OK)$  is the angle of parallelism of  $OK$  for  $\ell$ . The conclusion of the lemma is a well known property already established by Lobachevsky.  $\square$

**Lemma 9** *Let  $\alpha$  be a point at infinity. Let  $\delta_n^1$  and  $\delta_n^2$  be two rays issued from  $O$  such that  $O\alpha$  is the bisector of the angle  $(\delta_n^1, \delta_n^2)$  and  $(\delta_n^1, \delta_n^2) < \frac{\pi}{n}$ . Then, there is a unique line  $\ell_n$  of the hyperbolic plane such that  $\ell_n$  is parallel to both  $\delta_n^1$  and  $\delta_n^2$ . Let  $H_n$  be the half-plane defined by  $\ell_n$  which does not contain  $O$ . Then the  $H_n$ 's constitute a basis of neighbourhoods for  $\alpha$ .*

Proof. The existence of  $\ell_n$  is a well known property: it comes from the fact that  $\ell_n$  is the unique line of the hyperbolic plane which is parallel to  $\delta_1$  and which is perpendicular to  $O\alpha$ . In order to prove that the  $H_n$ 's constitute a basis of neighbourhoods for  $\alpha$ , we first note that a neighbourhood of  $\alpha$  is a subset of  $\mathbb{H}^2$  which contains a half-plane  $H$  which touches  $\alpha$ . Of course, we may assume that  $H$  does not contain  $O$ . Now, let  $\ell$  be the border of  $H$ . Consider the ray  $O\alpha$ : it cuts  $\ell$  at  $K$ , otherwise,  $H$  cannot touch  $\alpha$ . If it is perpendicular to  $\ell$ , there is a point  $L$  on  $O\alpha$  with  $[OKL]$  such that the parallel  $\ell_n$  issued from  $L$  to  $\delta_n^1$  is perpendicular to  $O\alpha$ . Then, as  $O\alpha$  is the bisector of  $(\delta_n^1, \delta_n^2)$ ,  $\ell_n$  is also parallel to  $\delta_n^2$ .

If  $O\alpha$  is not perpendicular to  $\ell$ , then there is a point  $L$  on  $O\alpha$  with  $[OKL]$  such that the perpendicular  $\mu$  to  $O\alpha$  passing through  $L$  is non-secant with  $\ell$ . We repeat with  $\mu$  the just above argument.

We remain to prove that if  $\beta$  is another point at infinity, so that  $\beta \neq \alpha$ , there is a  $H_n$  so that  $H_n$  does not touch  $\beta$ : we may even construct  $H_n$  so that its border  $\ell_n$  does not pass through  $\beta$ . Indeed, we take  $n$  so that  $\frac{1}{n} < (O\alpha, O\beta)$  and we repeat the above construction. It is then plain that  $H_n$  is contained in the half-plane delimited by  $O\delta_n^1$  which contains  $\alpha$ . By the construction, this



latter half-plane does not touch  $\beta$  as  $O\delta_n^1$  does not pass through  $\beta$  and as we may assume that  $\alpha$  and  $\beta$  are not on the same side of  $O\delta_n^1$ .  $\square$

Say that a line of the hyperbolic plane is a **line of the pentagrid** if it supports at least an edge of a pentagon of the tessellation. We wish to prove that in Lemma 9 we can replace the lines  $\ell_n$  by lines of the pentagrid. To this aim we prove the following result.

**Lemma 10** *Let  $\alpha$  be a point at infinity of the hyperbolic plane and let  $\ell$  be a line which does not pass through  $\alpha$ . Then there is a line of the pentagrid  $\lambda$  such that  $\lambda$  is completely contained in the half-plane defined by  $\ell$  which touches  $\alpha$ .*

*Proof.* Let  $b$  be the diameter of the regular rectangular pentagon. It is plain that  $a < b < \frac{5}{2}a$ : take any picture in Figure 1 to check the latter inequality as  $b$  is the distance from a vertex of the pentagon to the midpoint of the opposite side. This means that for any point  $P$  of the hyperbolic plane, within a disc of radius  $b$  centered at  $P$  we can find a vertex of the pentagrid. Consider  $\ell$ , a line of the hyperbolic plane which does not pass through  $\alpha$ . Denote by  $H_1$  the half-plane defined by  $\ell$  which touches  $\alpha$  and by  $H_2$  the other half-plane: that which does not touch  $\alpha$ . Take  $A$  a point on  $\ell$  and let  $S$  be a vertex of the pentagrid such that  $AS \leq b$ , which is in  $H_1$  and which is the closest to  $S$ . Let  $r_1$  and  $r_2$  be the rays issued from  $S$  which are supported by the lines of the pentagrid which meet at  $S$  and which delimit a quarter whose head  $P$  cuts  $\ell$ . If both  $r_1$  and  $r_2$  do not meet  $\ell$ , we are done. If we require  $r_1$  and  $r_2$  to be non-secant with  $\ell$ , we take  $R$  on the continuation of  $r_1$  in  $H_1$ , at the distance  $a$  and then we take  $T$  on the next side of the pentagon  $Q$  which contains  $R$  and  $S$  and which has a common side of  $P$ . Then the rays issued from  $T$  and supporting the edges of  $Q$  abutting  $T$  are non-secant with  $r_1$  and  $r_2$  as having a common perpendicular with these rays. At least one of the half-planes delimited by  $r_1$  and  $r_2$  touches  $\alpha$ . We take the line corresponding to this half-plane.

If  $r_1$  and  $r_2$  are not in this case, at least one of them, say  $r_1$  cuts  $\ell$ . Continue the ray  $r_1$  by the other ray  $y_1$  on the same line until we meet a vertex  $R$  of the pentagrid for which the other ray  $r_3$  abutting  $R$  and which is on the same side of  $r_1$  as  $P$ , is non-secant with  $\ell$ . Indeed, let  $K$  be the orthogonal projection of  $R$  on  $\ell$ . As  $R$  tends to infinity on  $y_1$ ,  $RK$  also tends to infinity and the angle of  $y_1$  with  $RK$  tends to zero so that we can find such an  $R$  that the angle of parallelism for  $RK$  with  $\ell$  is less than  $\frac{\pi}{4}$ . Then the angle  $\vartheta$  of  $y_1$  with  $RK$  satisfies  $\vartheta < \frac{\pi}{4}$  as  $r_1$  cuts  $\ell$ . Accordingly  $r_3$  makes an angle which is bigger than  $\frac{\pi}{4}$  so that  $r_3$  and its continuation in a line is non-secant with  $\ell$  and it clearly lies in  $H_1$ . Let  $y_3$  be the continuation of  $r_3$  after  $R$ . Then, we can find on  $y_3$  a vertex  $T$  of the pentagrid so that the perpendicular  $y_4$  to  $y_3$  passing through  $T$  is non-secant with  $\ell$ . Then at least one of the half-planes delimited by  $y_3$  and  $y_4$  and which does not contain  $\ell$  touches  $\alpha$ . We take the line defined by this half-plane.  $\square$

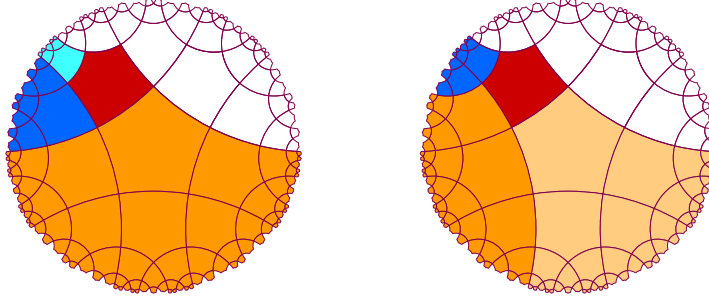
**Corollary 3** *The lines of the pentagrid define neighbourhoods for the points at infinity.*

*Proof.* It is a direct consequence of Lemmas 9 and 10.  $\square$

## 4 Preliminary properties

Figure 4 indicates two ways to decompose a quarter into other quarters.

Consider two quarters  $F_1$  and  $F_2$  whose vertices are  $S_1$  and  $S_2$  respectively and whose heads are  $H_1$  and  $H_2$  respectively. We say that  $F_1$  is **embedded**, **strictly embedded** in  $F_2$ , denoted by  $F_1 \sqsubseteq F_2$ ,  $F_1 \sqsubset F_2$  respectively, if  $F_1 \subseteq F_2$ ,  $F_1 \subset F_2^\circ$  respectively, where  $F_2^\circ$  is the interior of  $F_2$ . From the definition, strictly embedded quarters are embedded but embedded quarters may be not strictly embedded. Denote by  $\partial F$  the border of the quarter  $F$ . In the left-hand side of Figure 4, we can see that the orange quarter is embedded in the quarter  $Q$  whose head is the red tile. We also can see on the same picture that the blue quarter is strictly embedded in  $Q$ . On the right-hand side of Figure 4, the blue quarter and the quarter which extends the light orange zone are both embedded in  $Q$ , but not strictly. In the situation when the head of  $F_1$  shares an edge with the head of  $F_2$ , there are three possible cases. In two of them,  $F_1$  is embedded in  $F_2$  but not strictly, while in the third case,  $F_1$  is strictly embedded in  $F_2$ . We shall denote these cases by  $F_1 \sqsubseteq_0 F_2$  when the embedding is not strict and  $F_1 \sqsubset_0 F_2$  when the embedding is strict. The index 0 reminds us that the heads share an edge. In both cases we speak of a **one step** embedding. Note that when  $F_1 \sqsubset_0 F_2$ , it is not possible to find a quarter  $G$  such that  $F_1 \sqsubseteq_0 G$  and  $G \sqsubseteq_0 F_2$ . Now, we can prove the property indicated in Lemma 11.



**Figure 4** To left: the left-hand side decomposition.  
To right: the central decomposition.

**Lemma 11** *Let  $F_1$  and  $F_2$  be two embedded quarters whose vertices are  $S_1$  and  $S_2$  respectively. There is a finite sequence  $G_1, \dots, G_k$  of quarters such that  $F_1 = G_1$ ,  $F_2 = G_k$  and  $G_i \sqsubset_0 G_{i+1}$  or  $G_i \sqsubseteq_0 G_{i+1}$  for  $i \geq 1$  and  $i < k$ . Moreover, the distance from  $S_1$  to  $S_2$  is  $k-1$  in number of tiles.*

*Proof.* Identifying the head of  $F_2$  as the tile in bijection with the root of the Fibonacci tree, see the left-hand side picture of Figure 3, it is easy to find a finite sequence of tiles  $T_i$ , with  $i \in [1..k]$ , with  $T_1$  being the head of  $F_2$  and  $T_k$  that of  $F_1$ . Each tile is in correspondence with the nodes of the tree which are on the branch which leads from the root to the node in bijection with the head of  $F_1$ . By construction,  $T_1$  is the head of  $F_2$  and  $F_2 = G_k$  by construction. For each  $T_i$  with  $i > 1$ , we look at the place of  $T_i$  with respect to  $T_{i-1}$  which is the head of  $G_{k-i+1}$ . There are three possible cases only as indicated by Figure 4. If the edge shared with  $T_{i-1}$  has a vertex on the border of  $G_{k-i+2}$ ,

then we define  $G_{k-i+1}$  as indicated by Figure 4: there is a single possibility which yields  $G_{k-i+1} \sqsubseteq_0 G_{k-i+2}$ . If the edge shared with  $T_{i-1}$  has no vertex on the border of  $G_{k-i+2}$ , there is again a single possibility given by the left-hand side decomposition and we have  $G_{k-i+1} \sqsubset_0 G_{k-i+2}$ . The distance in number of tiles from  $S_1$  to  $S_2$  is the number of tiles on the branch, the last tile being excepted, so it is  $k-1$ .  $\square$

**Corollary 4** *Let  $F_1$  and  $F_2$  be two quarters whose vertices are  $S_1$  and  $S_2$  respectively. Then  $\text{dist}(S_1, S_2) > a$ .*

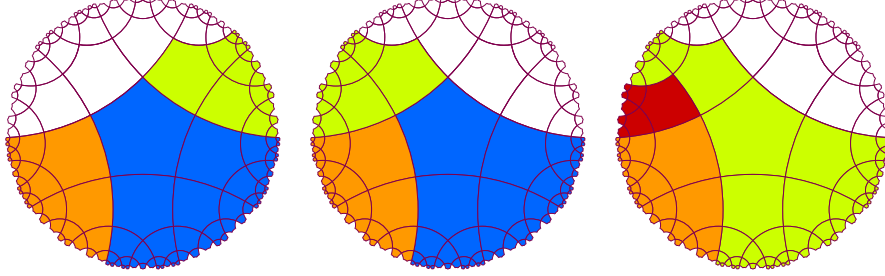
Proof. Clearly, the distance is bigger if the embedding is not in one step. For a one step embedding, Figure 4 clearly proves Corollary 4.  $\square$

## 5 Sequences of quarters

From what we have seen in Section 4, when we are dealing with a sequence  $\{F_n\}_{n \in \mathbb{N}}$  of quarters such that  $F_n \sqsubseteq F_{n+1}$ , we may assume that each embedding of consecutive terms of the sequence is a one step embedding. Say that such a sequence is **stepwise**.

Now, consider a stepwise sequence  $\{F_n\}_{n \in \mathbb{N}}$  of embedded quarters. Let  $S_n$  be the vertex of  $F_n$ . The sequence  $\{S_n\}_{n \in \mathbb{N}}$  cannot converge in the hyperbolic plane as the distance between two consecutive terms is at least  $a$ . Note that the topologies induced in Poincaré's disc by the Euclidean metric and by the hyperbolic one coincide despite the fact that the metrics are very different. This is a well known feature, coming from the property that hyperbolic circles are Euclidean circles contained in the open disc. Now, the closure of the disc is compact, so that the sequence  $\{S_n\}_{n \in \mathbb{N}}$  has at least one limit point  $\alpha$  which is a point of the border of the Poincaré's disc, which corresponds to an end of the hyperbolic plane.

Consider three consecutive terms of the sequence:  $F_n$ ,  $F_{n+1}$  and  $F_{n+2}$ . Consider the one-step relations between consecutive terms. If we have both  $F_n \sqsubseteq_0 F_{n+1}$  and  $F_{n+1} \sqsubseteq_0 F_{n+2}$ , we have two cases: we have either  $F_n \sqsubset F_{n+2}$  or  $F_n \not\sqsubset F_{n+2}$ , but in that latter case, we also have  $F_n \sqsubseteq F_{n+2}$ , see the first two pictures of Figure 5. The figure shows us that starting from  $F_{n+1}$ , there are two possibilities to construct  $F_{n+2}$  and only them: those which are illustrated by the pictures of the figure. Indeed, we have only two possibilities for choosing the new head. Once the new head is chosen, we have *a priori* two possibilities for choosing the vertex in order to obtain a quarter which contains  $F_{n+1}$ . But one of them defines  $F_{n+2}$  as strictly embedding  $F_{n+1}$ . So that a single vertex remains to obtain  $F_{n+2}$  as embedding  $F_{n+1}$  but not strictly, see the first two pictures of Figure 5. And so, we remain with the two cases which are illustrated by the first two pictures of Figure 5.



**Figure 5** The cases when  $F_n \sqsubseteq_0 F_{n+1}$  and  $F_{n+1} \sqsubseteq_0 F_{n+2}$ .  
To left: we have that  $F_n \sqsubseteq F_{n+2}$  and  $F_n \not\sqsubseteq F_{n+2}$ .  
Centre: we have that  $F_n \sqsubset F_{n+2}$ .  
To right: we have  $F_n \sqsubseteq_0 F'_{n+1}$  and  $F'_{n+1} \sqsubset_0 F_{n+2}$

### 5.1 Non-alternating sequences

Now, the rightmost picture of Figure 5 show us the following property:

**Lemma 12** Consider three quarters,  $F_1$ ,  $F_2$  and  $F_3$  such that  $F_1 \sqsubseteq_0 F_2$  and  $F_2 \sqsubset_0 F_3$ . Then, there is a quarter  $F_4$  such that:  $F_1 \sqsubseteq_0 F_4$  and  $F_4 \sqsubset_0 F_3$ . Conversely, if we have  $F_1 \sqsubseteq_0 F_2$ ,  $F_2 \sqsubseteq_0 F_3$  and  $F_1 \sqsubset F_3$ , we may find  $F_4$  such that  $F_1 \sqsubseteq_0 F_4$  and  $F_4 \sqsubset_0 F_3$ .

**Corollary 5** Let  $\{F_n\}_{n \in \mathbb{N}}$  be a stepwise sequence of consecutively embedded quarters. We may assume that if  $F_n \sqsubseteq_0 F_{n+1}$ ,  $F_{n+1} \sqsubseteq_0 F_{n+2}$  and  $F_n \sqsubset F_{n+2}$ , then we have  $F_{n+1} \sqsubset_0 F_{n+2}$ .

Consider a stepwise sequence of embedded quarters  $\{F_n\}_{n \in \mathbb{N}}$ . Say that  $F_{n+1}$  presents an **alternation** if and only if  $F_n \sqsubseteq_0 F_{n+1}$ ,  $F_{n+1} \sqsubseteq_0 F_{n+2}$  and  $F_n \sqsubset F_{n+2}$ . From Corollary 5, we may assume that a stepwise sequence  $\{F_n\}_{n \in \mathbb{N}}$  does not contain any alternation. This necessarily means that if  $F_n \sqsubset F_{n+2}$ , then  $F_{n+1} \sqsubset_0 F_{n+2}$ . We say that a stepwise sequence of embedded quarters  $\{F_n\}_{n \in \mathbb{N}}$  with no-alternation is **ultimately direct** if there is an integer  $N$  such that for all positive  $k$  we have  $F_N \not\sqsubseteq F_{N+k}$ . If in an ultimately direct sequence we may have  $N = 0$ , we say that the sequence is **direct**. We can state:

**Lemma 13** Let  $\{F_n\}_{n \in \mathbb{N}}$  be a stepwise sequence of embedded quarters with no alternation and assume the sequence to be ultimately direct. Let  $S_n$  be the vertex of  $F_n$ . Then  $S_n$  tends to the point at infinity  $\alpha$  which is on a line  $\ell$  which supports one border of all  $F_n$ 's starting from a certain rank. Moreover, all  $F_n$ 's are contained in the same half-plane defined by  $\ell$ .

**Proof.** From the assumption, we have an integer  $N$  such that  $F_n \sqsubseteq_0 F_{n+1}$  for all  $n \geq N$  and such that  $F_N \not\sqsubseteq F_{N+k}$  for any positive  $k$ . And so, there is a line  $\ell$  issued from  $S_N$  such that  $\ell$  contains a part of the border of  $F_N$  and such that for all positive  $k$ , there is a ray issued from  $S_{N+k}$  which is in the border of  $F_{N+k}$  and which is contained in  $\ell$ . Clearly,  $S_{N+k}$  converges to a point at infinity which is on  $\ell$  as  $k$  tends to infinity. Also clearly, as there is no alternation, all  $F_{N+k}$ 's

are on the same side of  $\ell$ . Due to the consecutive embedding of all terms of the sequence, all  $F_n$ 's are also in the same side.  $\square$

## 5.2 Limit of vertices

Now, what can be said for stepwise sequences of embedded quarters with no alternation which are not ultimately direct?

**Lemma 14** *Consider a stepwise sequence  $\{F_n\}_{n \in \mathbb{N}}$  of embedded quarters and assume it to be with no alternation and assume that the sequence is not ultimately direct. Let  $S_n$  be the vertex of  $F_n$ . Assume that the sequence  $\{S_n\}_{n \in \mathbb{N}}$  converges to an end  $\alpha$ . Let  $\ell$  be a line which does not pass through  $\alpha$ . Then, there is an  $N$  such that for all  $n$ ,  $n \geq N$ ,  $F_n$  contains the half-plane delimited by  $\ell$  which does not touch  $\alpha$ .*

Assuming Lemma 14, we can prove:

**Theorem 2** *Consider a sequence  $\{F_n\}_{n \in \mathbb{N}}$ . Let  $S_n$  be the vertex of  $F_n$ . Then there is an end  $\alpha$  such that  $S_n$  converges to  $\alpha$  when  $n$  tends to infinity.*

Proof of Theorem 2. From what we have already noticed, the sequence  $\{S_n\}_{n \in \mathbb{N}}$  has at least one limit point, and any limit point is an end. Assume that the sequence has at least two distinct limit points  $\alpha_1$  and  $\alpha_2$ . Then we can find lines  $\ell_1$  and  $\ell_2$  such that if  $\pi_1$  and  $\pi_2$  respectively are the half-planes defined by  $\ell_1$  and  $\ell_2$  and which touches  $\alpha_1$  and  $\alpha_2$  respectively, then  $\pi_1 \cap \pi_2 = \emptyset$ . Indeed, consider two lines  $m_1$  and  $m_2$  which pass by  $\alpha_1$  and  $\alpha_2$  respectively. As  $\alpha_1 \neq \alpha_2$ , the lines are distinct. We may assume that they meet at some point  $A$  of the hyperbolic plane. If not, the lines are non secant. Then replace  $m_1$  and  $m_2$  by the lines which are parallel to  $m_1$  and  $m_2$  and which are issued from the mid-point of the segment of the common perpendicular to  $m_1$  and  $m_2$  which joins  $m_1$  to  $m_2$ . From  $A$ , consider the bisector of the angle  $(A\alpha_1, A\alpha_2)$ . It defines a point at infinity  $\beta$ . Then take the bisector of  $(A\alpha_1, A\beta)$  and of  $(A\beta, A\alpha_2)$ . These new bisectors define two new points at infinity  $\beta_1$  and  $\beta_2$ . Now, define  $\ell_1, \ell_2$  as the perpendicular to  $m_1, m_2$  respectively, issued from  $\beta_1, \beta_2$  respectively.

Consider two sub-sequences of the  $F_n$ 's,  $\{G_k\}_{k \in \mathbb{N}}$  and  $\{H_k\}_{k \in \mathbb{N}}$  such the vertices of the  $G_k$  converge to  $\alpha_1$  and those of the  $H_k$  converge to  $\alpha_2$ . We have  $G_k = F_{n_k}$  and  $H_k = F_{m_k}$  where  $\{n_k\}_{k \in \mathbb{N}}$  and  $\{m_k\}_{k \in \mathbb{N}}$  are distinct sub-sequences of  $\mathbb{N}$ .

Assume that  $\{G_k\}_{k \in \mathbb{N}}$  and  $\{H_k\}_{k \in \mathbb{N}}$  are both ultimately direct. There is an integer  $N$  such that for all positive  $k$  we have both  $G_N \subseteq G_{N+k}$  and  $H_N \subseteq H_{N+k}$  together with both  $G_N \not\subseteq G_{N+k}$  and  $H_N \not\subseteq H_{N+k}$ . There is a line  $m_g$  and a line  $m_h$  such that  $m_g$  passes through  $\alpha_1$  and  $m_h$  passes through  $\alpha_2$  and, from Lemma 13, all  $G_k$ 's are on the same side of  $m_g$  and all  $H_k$ 's are on the same side of  $m_h$ . These sides define half-planes  $\mathcal{H}_g, \mathcal{H}_h$  delimited by  $m_g, m_h$  respectively. Assume that  $\mathcal{H}_h$  contains  $\mathcal{H}_g$  when we get close to  $\alpha_2$ . Consider some  $m$  with  $m > N$ . We can find  $n > m$  such that we have for instance that  $G_n$  contains  $H_m$ : but then,  $G_n$  contains points of the hyperbolic plane which are not in  $\mathcal{H}_g$ , a contradiction. So that we now assume that  $\mathcal{H}_h$  and  $\mathcal{H}_g$  do not meet when we get close to  $\alpha_2$ . But the same inclusion as above immediately shows that  $G_n$  contains points which are not in  $\mathcal{H}_g$ . Accordingly, in that case,  $\alpha_1 = \alpha_2$ .

Consider now that one  $\{G_k\}_{k \in \mathbb{N}}$  is ultimately direct, and that  $\{H_k\}_{k \in \mathbb{N}}$  is not. From Lemma 14 there is  $N$  such that when  $h \geq N$ ,  $H_h$  contains  $\mathbb{H}^2 \setminus \pi_2$ . Now, assume that  $\mathcal{H}_g \cap \pi_2 = \emptyset$ . Take a  $k > N$  such that  $n_k > m_h$ . Then,  $H_h$  contains  $\mathbb{H}^2 \setminus \pi_2$  as well as a few points of  $\pi_2$ . As  $n_k > m_h$ ,  $G_k$  contains also points in  $\pi_2$ , a contradiction as  $G_k \subset \mathcal{H}_g$ . Now, assume that  $\pi_2 \subset \mathcal{H}_g$ . Again, take  $k > N$  such that  $n_k > m_h$ . As  $G_k$  contains  $H_h$ , it also contains points which are on the complement of  $\mathcal{H}_g$ , again a contradiction. And so, in that case too,  $\alpha_1 = \alpha_2$ .

Now, we remain with the case when both sequences  $\{G_k\}_{k \in \mathbb{N}}$  and  $\{H_k\}_{k \in \mathbb{N}}$  are not ultimately direct. From Lemma 14 there is  $N$  such that when  $n \geq N$ ,  $H_n$  contains both  $\mathbb{H}^2 \setminus \pi_1$  and  $\mathbb{H}^2 \setminus \pi_2$ . But  $(\mathbb{H}^2 \setminus \pi_1) \cup (\mathbb{H}^2 \setminus \pi_2) = \mathbb{H}^2$ , so that  $F_m$  contains  $\mathbb{H}^2$  for a certain  $m$ , which is impossible. And so, we again conclude that  $\alpha_1 = \alpha_2$ . This proves that there is a unique limit point, hence the convergence of the sequence.  $\square$

### 5.3 Proof of Lemma 14

We can now turn to the proof of Lemma 14.

We already know that the sequence is stepwise, that it has no alternation, that it is not ultimately direct and that it has at least one limit point, say  $\alpha$ . From Lemma 10, we may replace  $\ell$  by a line  $\lambda$  of the pentagrid which does not pass through  $\alpha$ . Let  $H_1$  be the half-plane delimited by  $\lambda$  which touches  $\alpha$  and  $H_2$  be its complement in  $\mathbb{H}^2$ . It is also plain that if we find a quarter  $F_n$  satisfying the conclusion of the lemma, this will also be the case for all  $F_m$ 's with  $m \geq n$ .

There is a first  $n$  such the vertex  $S_n$  of the quarter  $F_n$  is in the interior of  $H_1$ . Accordingly, the head  $P$  of  $F_n$  has a side on  $\lambda$  and  $S_n$  is either one of its two vertices at the distance  $a$  from  $\lambda$  or the single one at the distance  $b$ . In the latter case we are done: the rays  $r_1$  and  $r_2$  issued from  $S_n$  have both a common perpendicular with  $\lambda$  so that  $F_n$  contains  $H_2$ .

Now, assume that  $S_n$  is at the distance  $a$  from  $\lambda$ . Let  $m$  be the first integer not smaller than  $n$  such that  $F_m \sqsubset_0 F_{m+1}$ . As there is no alternation,  $S_m$  is on the same line  $\ell_1$  passing through  $S_m$  and  $S_n$  which is perpendicular to  $\lambda$ . As the sequence is not ultimately direct, there is such an  $m$ . From the non-alternation assumption, the head  $P_{m+1}$  of  $F_{m+1}$  has one ray  $r_1$  of its border which is perpendicular to  $\ell_1$  and the other ray  $r_2$  is perpendicular to  $r_1$  and it lies in the same side of  $\ell_1$  as  $P_m$ . Then  $S_{m+1}$  is the vertex which is opposite to the side  $e$  of  $P_{m+1}$  shared with  $P_m$ . Now,  $\ell_1$  is a common perpendicular to  $r_1$  and to  $\lambda$ , so that  $r_1$  lies in  $H_1$ . Now,  $r_2$  has a common perpendicular with the line  $\mu$  which supports  $e$ . Now,  $\mu$  itself is perpendicular to  $\ell_1$ , so that it is contained in  $H_1$ . Accordingly,  $r_2$  is also contained in  $H_1$ , as it is on the other side of  $\mu$  with respect to  $\lambda$ . This proves that  $F_{m+1}$  contains  $H_2$ .  $\square$

## 6 Two non-computability results

Theorem 2 makes use of the compactness theorems which are not algorithmically true. We shall use the tools used in the proof of Theorem 2 to prove that it is algorithmically impossible to say whether two given ends are equal or not.

The theorem says that in a sequence of embedded quarters, their vertices converge to a limit which is a point at infinity of the hyperbolic plane. We can easily be convinced that a quarter can be clearly identified by three vertices of its head  $P$ : the vertex  $S$  of the quarter and the two vertices  $A$  and  $B$  of  $P$  which are joined to  $S$  by an edge of  $P$ . Call **hat of the quarter** the triple  $ASB$  or  $BSA$ . The rays defining the quarter are defined by  $SA$  and  $SB$  with  $S$  being the point from which the ray is issued and the second point being a point on the ray. As each vertex can be identified by a coordinate, see for instance [2, 3], the hat of a quarter is a piece of information which can easily be encoded for an algorithm. In an algorithmic approach, a sequence of embedded quarters is an algorithm, which, in principle, can also finitely be encoded. The embedding condition can also be encoded, much more easily if we assume the sequence to be stepwise with no-alternation. However, the fact that there is no alternation cannot algorithmically be checked and the stepwise condition also cannot algorithmically be checked: intuitively, this would require an infinite time. The algorithmic translation of Theorem 2 translates the sentence *to each sequence of embedded quarters, we can define an end to which the sequence of their vertices converge*. This notion of convergence means that it is possible to assign to each line  $\lambda$  of the pentagrid a rank  $N$  which ensures that the quarters with a higher rank are beyond  $\lambda$  and that the sequence of the  $\lambda$ 's define an end. We may assume that the lines of the pentagrid can also be encoded, for example, by a pair of vertices of the pentagrid. The convergence of this sequence of lines to an end cannot be checked but it nonetheless can be defined. Indeed, from Lemma 10, if a line  $\ell$  defines a half-plane containing an end  $\alpha$ , there is a line of the pentagrid  $\lambda$  which defines a half-plane also containing  $\alpha$ . This allows us to consider the ends which can be defined by a sequence of lines of the pentagrid.

And so, to each sequence of quarters, we associate a sequence of lines of the pentagrid which defines the end and this translation from a sequence of quarters to a sequence of lines of the pentagrid must be algorithmic. Assume also that two lines of the pentagrid being given, it is possible to decide whether they define non-intersecting half-planes or not. Now we show that it is not possible to algorithmically distinguish given ends.

**Theorem 3** *There is no algorithm which would for any sequence of lines of the pentagrid defining ends whether these ends are equal or not.*

*Proof.* The proof consists in constructing a sequence of sequences of quarters for which there is no algorithm defining an end. We define the sequence of sequences as follows. First, we need an algorithmic ingredient: it is the Kleene function,  $\mathcal{A}(m, n, k)$  which takes value 1 if the  $k^{\text{th}}$  step of computation of the  $m^{\text{th}}$  Turing machine halted on the data encoded by  $n$  and it takes value 0 if this is not the case. Note that if  $\mathcal{A}(m, n, k) = 1$ , then  $\mathcal{A}(m, n, k + 1) = 1$ .

Our algorithm works as follows. Fix a line of the pentagrid, say  $\delta_0$  which passes through  $O$ , a vertex of the pentagrid which we fixed once and for all. Fix  $\eta_0$  the other line of the pentagrid which passes through  $O$ . Define  $P_0$  to be a pentagon with vertex  $O$ . Call  $\alpha_0$  the end of  $\delta_0$  which is not in the same side as  $P_0$  with respect to  $\eta_0$ . Define  $A_0$  to be the other vertex of  $P_0$  on  $\eta_0$  and  $B_0$  to be the other vertex of  $P_0$  on  $\delta_0$ . The hat of  $F_0$  is then defined by the triple  $A_0OB_0$ . For each  $n$  and  $k$ , we define a quarter of  $F_{n,k}$  by its head  $P_{n,k}$  and its hat:  $A_{n,k}S_{n,k}B_{n,k}$ . Define  $P_{0,0} = P_0$ ,  $S_{0,0} = O$ ,  $A_{0,0} = A_0$ ,  $B_{0,0} = B_0$  and  $S_{n,-1} = B_0$ . We define a flag by  $f = 0$ .

- As long as  $\mathcal{A}(n, n, k+1) = f$ ,  $P_{n,k+1}$  is the reflection of  $P_{n,k}$  in  $S_{n,k}A_{n,k}$ ;  $S_{n,k+1}$  is the reflection of  $S_{n,k-1}$  in  $S_{n,k}A_{n,k}$  too,  $A_{n,k+1}$  is the other end of the side of  $P_{n,k+1}$  which passes through  $S_{n,k+1}$  and which is orthogonal to  $\delta_f$ ,  $B_{n,k+1}$  is  $S_{n,k}$ .
- If  $\mathcal{A}(n, n, k+1) = 1$  and  $f = 0$ , then  $f := 1$ ;  $P_{n,k+1}$  is still the reflection of  $P_{n,k}$  in  $S_{n,k}A_{n,k}$ ;  $A_{n,k+1}$  is the vertex of  $P_{n,k+1}$  which is the reflection of  $B_{n,k-1}^\ell$  in  $S_{n,k}A_{n,k}$ ,  $S_{n,k+1}$  is the other end of the side of  $P_{n,k+1}$  which passes through  $A_{n,k+1}$  and which is orthogonal to  $\delta_0$ ,  $B_{n,k+1}$  is the other end of the side of  $P_{n,k+1}$  which ends at  $S_{n,k+1}$  and which does not meet  $\delta_0$ ,  $S_{n,k-1} = B_{n,k+1}$ , let  $\delta_1$  be the line defined by  $S_{n,k+1}B_{n,k+1}$ .

It is clear that for each  $n$ , the sequence  $\{F_{n,k}\}_{k \in \mathbb{N}}$  is a sequence of embedded quarters. The sequence is stepwise and it has no alternation by construction. If the Turing machine numbered by  $n$  does not halt on the data  $n$ , then  $\mathcal{A}(n, n, k) = 0$  for all  $k$  and so, the sequence is ultimately direct, it is even direct, so that, by Lemma 13, the sequence  $S_{n,k}$  converges to  $\alpha_0$ . If the Turing machine numbered by  $n$  halts on the data  $n$ , there is an integer  $m$  such that  $\mathcal{A}(n, n, m) = 0$  and  $\mathcal{A}(n, n, m+1) = 1$ . Accordingly,  $F_{n,m} \sqsubset_0 F_{n,m+1}$  but, afterwards, the sequence satisfies  $F_{n,k} \sqsubseteq_0 F_{n,k+1}$  and  $F_{n,k} \not\sqsubset_0 F_{n,k+1}$ . The sequence is again ultimately alternate but, this time, it converges to the end  $\alpha_1$  of  $\delta_1$  which is contained in the other side of  $\eta_0$  with respect to  $P_0$ . Now, By construction, as  $S_{n,m+1}A_{n,m+1}$  is a common perpendicular to  $\delta_0$  and  $\delta_1$ , these lines are non-secant, in particular, they cannot be parallel. Accordingly,  $\alpha_0 \neq \alpha_1$ .

Now, if  $\alpha_0 \neq \alpha_1$ , among the lines of the pentagrid which defines these ends, we can find two of them  $\lambda_1$  and  $\lambda_2$  such that denoting by  $\pi_1, \pi_2$  the half-plane defined by  $\lambda_1, \lambda_2$  respectively and which touches  $\alpha_1, \alpha_2$  respectively, we get  $\pi_1 \cap \pi_2 = \emptyset$ . And this can be performed algorithmically if  $\alpha_0 \neq \alpha_1$ . Now, if we had an algorithm which could tell us whether these limits are the same or not, this algorithm could be used to decide the halting problem for Turing machines, which is known to be impossible.  $\square$

Now, we can prove another result of the same flavor.

**Theorem 4** *We can construct a sequence  $\{G_{k,n}\}_{k \in \mathbb{N}}$  of quarters whose vertices are  $S_{k,n}$  such that for each  $n$  the sequence  $\{S_{k,n}\}_{k \in \mathbb{N}}$  converges to a point at infinity  $y_n$  and such that the sequence  $\{y_n\}_{n \in \mathbb{N}}$  cannot algorithmically converge to any point at infinity.*

Note that if the sequence  $\{y_n\}_{n \in \mathbb{N}}$  converges, it must converge to a point at infinity.

Proof of Theorem 4. Consider the same function  $\mathcal{A}(n, n, m)$  as previously. We change the construction as follows. We construct a sequence of sequences  $\{G_{m,n}\}_{m \in \mathbb{N}}$ , again defining a quarter by its head and its hat. In what follows, the hat will be given as previously but the order of the vertices is important. In a tile  $P_{k,n}$ , we consider that the hat is  $A_{k,n}S_{k,n}B_{k,n}$ . Let  $e_{k,n}$  be the side which is opposite to  $S_{k,n}$ . We consider that  $e_{k,n}$  is at the **bottom** of the tile, that  $S_{k,n}$  is at its top, that  $A_{k,n}$  is at its left-hand side and  $B_{k,n}$  at its right hand-side: we can consider that starting from  $A_{k,n}$  and clockwise turning around the tile we meet  $S_{k,n}$ , then  $B_{k,n}$  and then the ends of  $e_{k,n}$ . The side  $A_{k,n}S_{k,n}$  will be called side 0 and the side  $S_{k,n}B_{k,n}$  will be called side 1. For each fixed  $n$ , we start with a fixed once and for all tile with bottom  $e_0$  and hat  $A_0S_0B_0$  which will be denoted by  $P_{0,n}$ . At the beginning  $k = 0$ . Then we construct the sequence as follows.



- If  $\mathcal{A}(k+1, k+1, n) = 0$ ,  $P_{k+1,n}$  is the reflection of  $P_{k,n}$  in  $A_{k,n}S_{k,n}$ ;  $e_{k+1,n}$  is  $A_{k+1,n}S_{k+1,n}$ , which fixes the hat with the conventions we have already defined. We say that  $P_{k+1,n}$  has the value 0.
- If  $\mathcal{A}(k+1, k+1, n) = 1$ , then  $P_{k+1,n}$  is the reflection of  $P_{k,n}$  in  $S_{k,n}B_{k,n}$ ;  $e_{k+1,n}$  is  $S_{k,n}B_{k,n}$ . We say that  $P_{k+1,n}$  has the value 1.

$G_{k,n}$  is the quarter defined by  $P_{k,n}$  and its hat  $A_{k,n}S_{k,n}B_{k,n}$ . By construction, it is plain that for each  $n$  and  $k$  we have  $G_{k,n} \sqsubset_0 G_{k+1,n}$ . Accordingly, the sequence  $G_{k,n} \sqsubset_0 G_{k+1,n}$  is stepwise, with no alternation and it is not ultimately direct: when  $n$  is fixed, there is always a Turing machine numbered with  $k$  such that its computation on  $k$  is completed at the  $n^{\text{th}}$  step. From Theorem 2, the sequence  $\{S_{k,n}\}_{k \in \mathbb{N}}$  tends to a point at infinity  $y_n$ . By construction of the quarters, we can notice that for each  $k$  the rest of the sequence evolves in  $\mathbb{H}^2 \setminus G_{k,n}$ . Fix  $k$  and let  $K_0 = \mathbb{H}^2 \setminus G_{k,n}$  when  $\mathcal{A}(k, k, n) = 0$  and  $K_1 = \mathbb{H}^2 \setminus G_{k,n}$  when  $\mathcal{A}(k, k, n) = 1$ . It is not difficult to see that  $K_0 \cap K_1 = \emptyset$ . More than that, the line defined by  $S_{k,n}B_{k,n}$  for  $K_0$  and the line defined by  $A_{k,n}S_{k,n}$  for  $K_1$  are non-secant. This means that the distance between  $K_0$  and  $K_1$  tends to infinity when we go to infinity on both these borders. From this remark, assume that the sequence  $\{y_n\}_{n \in \mathbb{N}}$  tends to a limit  $y$  which is also a point at infinity. Then there is a half-plane  $H$  delimited by a line  $\lambda$  which may be assumed to belong to the pentagrid such that there is  $N$  such that for  $n \geq N$ , all  $y_n$ 's are touched by  $H$ . By the remark we made about  $K_0$  and  $K_1$ , we can see that, necessarily, for any  $n \geq N$ ,  $G_{k,n} = G_{k,N}$ , otherwise,  $y_n$  and  $y_N$  cannot be both in  $H$ . Accordingly, if we have an algorithm  $\varphi$  which, for each  $n$ , gives an integer  $\varphi(n)$  such that all  $y_p$  with  $p \geq \varphi(n)$  are in  $H$  which is at distance  $n$  from  $P_0$ , then, looking at the value of  $G_{n,\varphi(n)}$ , we know whether  $\mathcal{A}(k, k, n) = 0$  for ever or not. And this decides the halting problem, which is impossible.  $\square$

## 7 conclusion

Probably, other undecidability results of analysis can be transported into the hyperbolic plane in similar way. This might open a new area.

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